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LETTER TO THE EDITOR

A new method of determining an irreducible representation of quantum $\mathrm{Sl}_q(3)$ algebra (I)

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Abstract. The explicit forms of the irreducible representation matrices for the quantum $\mathrm{Sl}_q(3)$ enveloping algebra are computed by a new technique.

Recently there has been an increasing interest in quantum enveloping algebra for both physicists and mathematicians. Much literature on the subject has appeared (Jimbo 1985, 1986, Biedenharn 1989, Ng 1990). In order to apply the quantum enveloping algebra in physics it is necessary to develop their representations systematically. In this letter, we will suggest a new method of constructing all the irreducible representations of $\mathrm{Sl}_q(3)$ algebra explicitly. The method presented in this letter can easily be used to calculate their Clebsch–Gordan coefficients (q-CGC) and to generalize to quantum $\mathrm{Sl}_q(n)$ algebra without any difficulties.

The general relations of quantum $\mathrm{Sl}_q(3)$ enveloping algebra are given by Jimbo (Jimbo 1985, Song 1990) as follows

$$[h_a, e_{\pm a}] = \pm 2e_{\pm a} \qquad a = 1, 2 \tag{1a}$$

$$[h_a, e_{\pm b}] = \mp e_{\pm b} \qquad a \neq b, a, b = 1, 2 \tag{1b}$$

$$[e_a, e_{-a}] = [h_a] \qquad a = 1, 2 \tag{1c}$$

$$[h_a, e_{\pm 3}] = \pm e_{\pm 3} \qquad a = 1, 2 \tag{1d}$$

$$[e_3, e_{-3}] = h_1 + h_2 \tag{1e}$$

and

$$e_a^2 e_b + e_b e_a^2 = [2] e_a e_b e_a \qquad a \neq b, a, b = 1, 2 \text{ or } -1, -2 \tag{2a}$$

$$e_{\mp}^2 e_{\pm 3} + e_{\pm 3} e_{\mp 1}^2 = [2] e_{\mp 1} e_{\pm 3} e_{\mp 1} \tag{2b}$$

where

$$e_a^+ = e_{-a} \qquad a = 1, 2, 3 \tag{3a}$$

$$h_a^+ = h_a \qquad a = 1, 2 \tag{3b}$$

and

$$[x] = (q^x - q^{-x})/(q - q^{-1}). \tag{4}$$

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For attending our aim, we define

$$J_0 = h_1/2 \qquad J_\pm = e_{\pm 1} \tag{5a}$$

$$Q_0 = -(h_1 + h_2) \tag{5b}$$

and

$$T_{1/2} = -e_{-2} \qquad T_{-1/2} = e_{-3} \qquad V_{-1/2} = e_2 \qquad V_{1/2} = e_3. \tag{5c}$$

Obviously

$$V_s = (-1)^{1/2-s} T^+_{-s} \qquad s = \pm 1/2. \tag{6}$$

Now the algebra relations of (1) become

$$[Q_0, J_0] = [Q_0, J_\pm] = 0 \tag{7a}$$

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0] \tag{7b}$$

$$[J_0, T_{\pm 1/2}] = \pm\tfrac{1}{2} T_{\pm 1/2} \qquad [J_0, V_{\pm 1/2}] = \pm\tfrac{1}{2} V_{\pm 1/2} \tag{7c}$$

$$[Q_0, T_{\pm 1/2}] = 3\, T_{\pm 1/2} \qquad [Q_0, V_{\pm 1/2}] = -3\, V_{\pm 1/2} \tag{7d}$$

and

$$J_-^2 T_{1/2} + T_{1/2} J_-^2 = [2] J_- T_{1/2} J_- \tag{8a}$$

$$J_+^2 T_{-1/2} + T_{-1/2} J_+^2 = [2] J_+ T_{-1/2} J_+ \tag{8b}$$

etc. That is to say the operators J_0, $J_\pm$ form the quantum $\mathrm{Sl}_q(2)$ algebra.

Due to the similarity between (7) and the corresponding relations of the classical algebra Su(3) (Sun 1965), we can take the Elliott-like wavefunction $|(\lambda\mu)\varepsilon JM\rangle$ as the set of basic vectors of algebra $\mathrm{Sl}_q(3)$,

$$Q_0 |(\lambda\mu)\varepsilon JM\rangle = \varepsilon |(\lambda\mu)\varepsilon JM\rangle \tag{9a}$$

$$J^2 |(\lambda\mu)\varepsilon JM\rangle = [J][J+1] |(\lambda\mu)\varepsilon JM\rangle \tag{9b}$$

$$J_0 |(\lambda\mu)\varepsilon JM\rangle = M |(\lambda\mu)\varepsilon JM\rangle. \tag{9c}$$

Here J^2 is the Casimir operator of $\mathrm{Sl}_q(2)$ algebra (Curtright 1990)

$$J_2 = \begin{cases} J_- J_+ + [J_0 + \frac{1}{2}]^2 & J = \frac{1}{2}\ \text{integer} \qquad (10a) \\ J_- J_+ + [J_0][J_0 + 1] & J = \text{integer}. \qquad (10b) \end{cases}$$

The quantum numbers λ and μ which will be determined below label an irreducible representation of $\mathrm{Sl}_q(3)$.

According to conventionality, the phase factors between $|(\lambda\mu)\varepsilon JM\rangle$ and $|(\lambda\mu)\varepsilon JM \pm 1\rangle$ are fixed by

$$\langle (\lambda\mu)\varepsilon' J' M' | J_\pm | (\lambda\mu)\varepsilon JM\rangle = \sqrt{[J \mp M][J \pm M + 1]}\, \delta_{\varepsilon'\varepsilon} \delta_{J'J} \delta_{M'M\pm 1}. \tag{11}$$

Now we turn to calculate the representation matrices of the operators T_s and V_s. For simplicity we will omit the quantum numbers λ and μ for the time being.

From (8), we have

$$\begin{aligned} &\sqrt{[J'-M'][J'+M'+1][J'-M'-1][J'+M'+2]}\langle \varepsilon + 3J'M'+2 | T_{1/2} | \varepsilon JM\rangle \\ &\quad + \sqrt{[J+M][J-M+1][J+M-1][J-M+2]}\langle \varepsilon + 3J'M' | T_{1/2} | \varepsilon JM - 2\rangle \\ &= [2]\sqrt{[J'-M'][J'+M'+1][J+M][J-M+1]} \\ &\quad \times \langle \varepsilon + 3J'M'+1 | T_{1/2} | JM\rangle \end{aligned} \tag{12a}$$

$$\sqrt{[J'+M'][J'-M'+1][J'+M'-1][J'-M'+2]}\langle \varepsilon+3J'M'-2|T_{-1/2}|\varepsilon JM\rangle$$
$$+\sqrt{[J-M][J+M+1][J-M-1][J+M+2]}\langle \varepsilon+3J'M'|T_{-1/2}|\varepsilon JM+2\rangle$$
$$=[2]\sqrt{[J'+M'][J'-M'+1][J-M][J+M+1]}$$
$$\times\langle \varepsilon+3J'M'-1|T_{-1/2}|\varepsilon JM+1\rangle. \qquad (12b)$$

From $(7c)$, obviously $M'=M-3/2$ and $M'=M+3/2$ in $(12a, b)$ respectively. Solving (12), we obtain

$$\langle \varepsilon+3J'M'|T_s|\varepsilon JM\rangle = F(\varepsilon, J', J)f(J', M', J, M) \qquad (13)$$

where

$$f(J, M, J, M)=\begin{cases}\sqrt{\dfrac{[J\pm M'+1/2]}{[2J+1]}} & J'=J\pm 1/2,\ s=1/2\\[2ex] \sqrt{\dfrac{[J\mp M'+1/2]}{[2J+1]}} & J'=J\pm 1/2,\ s=1/2.\end{cases} \qquad (14)$$

From (14), we can rewrite $F(\varepsilon, J', J)=\langle \varepsilon+3J'\|T\|\varepsilon J\rangle/\sqrt{[2J+1]}$, $f(J', M', J, M)=q^{A/2}C_q(JM\tfrac{1}{2}s\,|\,J'M')$ and

$$\langle \varepsilon+3J'M'|T_s|\varepsilon JM\rangle=\frac{\langle \varepsilon+3J\|T\|\varepsilon J\rangle}{[2J'+1]}\,q^{A/2}C_q(JM, 1/2s|J'M')$$

$$A=\begin{cases}(-1)^{\frac{1}{2}+J-J'}(J-M'+1/2) & J'=J\pm 1/2,\ s=1/2\\ (-1)^{\frac{1}{2}+J'-J}(J+M'+1/2) & J'=J\pm 1/2,\ s=-1/2\end{cases} \qquad (15)$$

where $C_q(JM, 1/2s|J'M')$ is the CGC of $\mathrm{Sl}_q(2)$ (Hou *et al* 1990). Equation (15) can be considered as the quantum Wingner–Eckart theorem and $\langle \varepsilon+3J'\|T\|\varepsilon J\rangle$ is the reduced matrix elements of the operator T_s. From (15), (6) and the symmetry properties of the $\mathrm{Sl}_q(2)$ CGC, we have

$$\langle \varepsilon-3J'\|V\|\varepsilon J\rangle=(-1)^{\frac{1}{2}+J-J'}\langle \varepsilon J\|T\|\varepsilon-3J'\rangle. \qquad (16)$$

According to $(1c)$ and $(1e)$, the commutation relations of the operators T_s and V_s can be obtained as follows

$$[T_{1/2}, V_{-1/2}]=-[Q_0/2+J_0] \qquad (17a)$$
$$[T_{-1/2}, V_{1/2}]=[Q_0/2-J_0]. \qquad (17b)$$

And using (2), we give out the recursion formulae of the reduced matrix elements

$$|\langle \varepsilon J\|T\|\varepsilon-3J+1/2\rangle|^2$$
$$=[2J+2][\varepsilon/2-J]+1/[2J+1]|\langle \varepsilon+3J+1/2\|T\|\varepsilon J\rangle|^2$$
$$+[2J+2]/[2J+1]|\langle \varepsilon+3J-1/2\|T\|\varepsilon J\rangle|^2 \qquad (18a)$$

$$|\langle \varepsilon J\|T\|\varepsilon-3J-1/2\rangle|^2$$
$$=[2J][\varepsilon/2+J+1]+[2J]/[2J+1]|\langle \varepsilon+3J+1/2\|T\|\varepsilon J\rangle|^2$$
$$-/[2J+1]|\langle \varepsilon+3J-1/2\|T\|\varepsilon J\rangle|^2. \qquad (18b)$$

Due to

$$T_s|(\lambda\mu)\varepsilon_{\max}J_0M\rangle=0 \qquad s=\pm 1/2 \qquad (19)$$

the $|(\lambda\mu)\varepsilon_{\max}J_0M\rangle$ is called the highest weight state. Using the mathematical inductive method, we can prove that

$$|\langle\varepsilon_{\max}-3nJ_0+n/2-i\|T\|\varepsilon_{\max}-3n-3J_0+(n+1)/2-i\rangle|^2$$
$$=[1+n-i][2J+2+n-i][\varepsilon_{\max}/2-J_0-n+i] \quad (20a)$$
$$|\langle\varepsilon_{\max}-3nJ_0+n/2-i\|T\|\varepsilon_{\max}-3n-3J_0+(n-1)/2-i\rangle|^2$$
$$=[1+i][2J-i][\varepsilon_{\max}/2+J_0+1-i] \quad (20b)$$

where $n, i=0,1,2,3,\ldots$. From the above formulae we can show that the quantum numbers in the highest weight satisfy

$$\varepsilon_{\max}=2\lambda+\mu \qquad J_0=\mu/2. \quad (21a)$$

Similarly we can also obtain

$$\varepsilon_{\min}=-\lambda-2\mu \qquad J_0'=\lambda/2. \quad (21b)$$

Choosing the phase facts from among the wavefunctions $|(\lambda\mu)\varepsilon JM\rangle$ in order to ensure that the reduced matrix elements $\langle\varepsilon+3J'\|T\|\varepsilon J\rangle$ are real and positive, then all values of the $\langle\varepsilon+3J'\|T\|\varepsilon J\rangle$ and $\langle\varepsilon-3J'\|V\|\varepsilon J\rangle$ can be obtained. Having known these values, all the wavefunctions $|(\lambda\mu)\varepsilon JM\rangle$ may be easily calculated, since

$$J_\pm|(\lambda\mu)\varepsilon JM\rangle=\sqrt{[J\mp M][J\pm M+1]}\,|(\lambda\mu)\varepsilon JM\pm1\rangle \quad (22a)$$
$$|(\lambda\mu)\varepsilon-3J'M'\rangle=(-1)^{\frac{1}{2}+J-J'}N(\varepsilon J'J)\sum_{MS}C_q(JM1/2S|J'M')\,V_s|(\lambda\mu)\varepsilon JM\rangle \quad (22b)$$

where $N(\varepsilon J'J)$ is a normalized constant,

$$\{N(\varepsilon J'J)\}^{-1}=\langle\varepsilon J\|T\|\varepsilon-3J'\rangle/\sqrt{[2J'+1]}\sum_{MS}\{C_a(JM1/2s|J'M')\}q^A. \quad (23)$$

By analogy with the classical algebra su(3), we can also find out the correlation between the Elliott basis and the Gelfand basis (Li and Song 1990).

The CGC of $\mathrm{Sl}_q(3)$ will appear in a future publication.

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References

Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Curtright T L 1990 *Phys. Lett.* **243B** 237
Hou B-Y, Hou B-Y and Ma Z-Q 1990 *Commun. Theor. Phys.* **13** 181; 341
Jimbo M 1985 *Lett. Math. Phys.* **10** 63
—— 1986 *Lett. Math. Phys.* **11** 247
Li L and Song X-C 1990 *Commun. Theor. Phys.* **13** 209
Ng Y J 1990 *J. Phys. A: Math. Gen.* **23** 1023
Song X-C 1990 *J. Phys. A: Math. Gen.* **23** L821
Sun H C 1965 *Scientia Sinica* **14** 840